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Modules over Pullbacks and Subdirect Sums*

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Let $f_1: R_1 \twoheadrightarrow \bar{R}$ and $f_2: R_2 \twoheadrightarrow \bar{R}$ be homomorphisms of two rings R_i onto a common ring $\bar{R}$, and let R be the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 \mid f_1(r_1) = f_2(r_2)\}$. Can all R-modules be described as some kind of combination of modules over R_1, R_2, and $\bar{R}$?

We answer this question—in fact, an n-coordinate version of it—when the "combining" ring $\bar{R}$ is semisimple artinian, and the coordinate rings R_i are completely arbitrary.

Very briefly: It is *not* true that an arbitrary R-module M is a pullback of modules over the coordinate rings. However, there is an epimorphism $\varphi: S \twoheadrightarrow M$ of R-modules, where S *is* such a pullback, and where one cannot get any "closer to M" by a pullback of this type. S turns out to be unique up to isomorphism, in fact, up to isomorphism over M. We call such a homomorphism φ a *separated representation* of M.

The main results are strong enough to enable us to give a complete description of all finitely generated modules over (i) the integral group ring $\mathbb{Z}G$, G cyclic of prime order (including those ZG-modules whose abelian group is torsion or mixed), and (ii) any subring of prime index in $\mathbb{Z} \oplus \mathbb{Z}$. (Modules over these latter rings can be surprisingly complicated.) The details of these applications will be found in [1]. Here we develop only the existence and uniqueness of separated representations, but in the much more general context outlined above.

1. DEFINITIONS AND MAIN RESULTS

Let the ring R be a *subdirect sum of* $R_1 \oplus \cdots \oplus R_n$, by which we mean that $R \subseteq R_1 \oplus \cdots \oplus R_n$ and each projection $R \to R_i$ maps R *onto* R_i. $R \cap R_i$ will mean the set of elements $r = (r_1, \ldots, r_n)$ of R such that $r_j = 0$ except

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possibly for $j=i$. Thus $R \cap R_i$ is an ideal of both R and R_i, and the sum $\sum_i (R \cap R_i)$ is a direct sum. Let

$$\bar{R} = R \Big/ \bigoplus_{i=1}^{n} (R \cap R_i).$$

The above notation *will be fixed* throughout this section and the next. In addition, we will usually suppose that $\bar{R}$ is *semisimple artinian.*

In the extreme case $\bar{R}=0$, we have that $R \supseteq$ every R_i (since the coordinate projections $R \to R_i$ are onto); in simpler language: $R = R_1 \oplus \cdots \oplus R_n$. Thus we may think of the fact that ${}_R\bar{R}$ has finite composition length as specifying that R contains "most" of the direct sum $R_1 \oplus \cdots \oplus R_n$.

When R is the pullback of $R_1 \oplus R_2$ mentioned at the very beginning of this paper, it is easy to see that $R \cap R_1 = (\ker f_1, 0)$ and $R \cap R_2 = (0, \ker f_2)$. To see that the two rings called $\bar{R}$ are isomorphic, just note that the kernel of the map: $R \twoheadrightarrow$ (the combining ring $\bar{R}$ of the previous section) given by $(r_1, r_2) \to f_1(r_1)$ $(=f_2(r_2))$ is $R \cap R_1 \oplus R \cap R_2$; hence $\bar{R} \cong R/\oplus(R \cap R_i)$.

We will call an R-module S *separated* if S is an R-submodule of a direct sum $S_1 \oplus \cdots \oplus S_n$ where each S_i is an R_i-module [made into an R-module by $(r_1,\dots, r_n)(s_1,\dots, s_n) = (r_1 s_1,\dots, r_n s_n)$]. We caution the reader that the isomorphism classes of the coordinate modules S_i are *not uniquely* determined by S, even when S is a subdirect sum of $S_1 \oplus \cdots \oplus S_n$ (that is, even when the projections $S \to S_i$ are onto). However, we will see, in Corollary 3.3, that the coordinate modules can always be chosen *canonically*, and that S has the structure of a pullback, very similar to that of R.

MAIN DEFINITION. A *separated representation* of an R-module M (R as at the beginning of this section) is an R-module epimorphism $\varphi\colon S \twoheadrightarrow M$ such that

S is a separated R-module; and (1)

S is "as close as possible" to M in the sense that if φ admits a factorization

$$\varphi\colon S \xrightarrow{f} S' \twoheadrightarrow M \tag{2}$$

with S' also a separated R-module, then f must be one-to-one.

Main Results

We show that every R-module (finitely generated *or not*) M has a separated representation $\varphi\colon S \twoheadrightarrow M$. If also $\varphi'\colon S' \twoheadrightarrow M$ is a separated representation, then there is an isomorphism $f^*: S' \twoheadrightarrow S$ such that $\varphi' = \varphi f^*$ (Theorem 2.8).

The most interesting property of separated representations is their "almost functorial" property: any homomorphism of R-modules can be lifted to a homomorphism of their separated representations (Theorem 2.6). This can be regarded as the main theorem of Section 2; almost everything else either is needed in its proof or follows easily from it. It is also the main tool used in the applications to be given in [1].

We also obtain two minimality properties of separated representations $\varphi: S \twoheadrightarrow M$. First is that no submodule strictly less than S is mapped, by φ, *onto* M (Proposition 2.5). Second is that if $\varphi': S' \twoheadrightarrow M$ is any R-module epimorphism, with S' separated, then there is a factorization

$$\varphi': S' \longrightarrow\!\!\!\!\rightarrow S \xrightarrow{\varphi}\!\!\!\!\rightarrow M.$$

This research was done with the case $n = 2$ specifically in mind, in order to describe modules over the integral group ring $\mathbb{Z}G$, G cyclic of prime order. However, all of the proofs of Section 2 can all be done for n coordinates instead of two coordinates by merely writing n in place of 2. Thus one wonders what kind of subdirect sum we are dealing with, in this more general case. In Section 3 we show that R is what we call a "multiple pullback."

2. Separated Representations

Throughout this section, let R, $R_1 \oplus \cdots \oplus R_n$, and $\bar{R}$ be as in the first paragraph of Section 1, *with $\bar{R}$ semisimple artinian.*

LEMMA 2.1. *Let S be a separated R-module. Then*

(i) *The sum $\sum (R \cap R_i) S$ is a direct sum.*

(ii) *If T is an R-submodule of S, and T is an internal direct sum $T = \oplus_{i=1}^{n} T_i$ with each $T_i \subseteq (R \cap R_i)S$, then S/T is again a separated R-module.*

Proof. Let $S \subseteq S_1 \oplus \cdots \oplus S_n$ with each S_i an R_i-module. Then $(R \cap R_i)S \subseteq S_i$, so (i) follows. To obtain (ii), note that the kernel of the map $S \to \oplus S_i/T_i$ given by

$$(s_1, \ldots, s_n) \to (s_1 + T_1, \ldots, s_n + T_n)$$

is T, so S/T can be considered to be a submodule of $\oplus S_i/T_i$. Moreover S_i/T_i is an R_i-module because S_i is.

LEMMA 2.2. *Let U be a submodule of an R-module S, and suppose $U \cap \ker(R \to \bar{R})S = 0$. Then U is a direct summand of S.*

Proof. Let ν be the natural homomorphism of S onto $\bar{S} = S/\ker(R \to \bar{R})S$. Since $\bar{S}$ is an $\bar{R}$-module, and $\bar{R}$ is semisimple artinian, $\nu(U)$ is a direct summand of $\bar{S}$. Let π be a projection map of $\bar{S}$ onto $\nu(U)$. By hypothesis, the restriction $\nu \mid U$ of ν to U is one-to-one; so we can form the map $(\nu \mid U)^{-1}\pi\nu: S \to U$, which is the identity on U.

Thus we have proved the lemma by displaying a projection map of S onto U.

PROPOSITION 2.3. *Let $\varphi: S \twoheadrightarrow M$ be an R-module epimorphism, with S a separated module. Then the following statements are equivalent.*

(i) *$\varphi: S \twoheadrightarrow M$ is a separated representation of M.*

(ii) *φ is one-to-one on $(R \cap R_i)S$ for each i, and* $\ker \varphi \subseteq \ker(R \to \bar{R})S$.

(iii) *φ is one-to-one on $(R \cap R_i)S$ for each i, and no nonzero direct summand of S is contained in* $\ker \varphi$.

Proof. (i) $\Rightarrow$ (iii). Let $S = T \oplus K$, with $K \subseteq \ker \varphi$. Then there is a factorization $\varphi: S \to T \twoheadrightarrow M$ with T separated, because every submodule of a separated module is again separated. So $S \to T$ is one-to-one. Thus $K = 0$ as desired.

Next, let $K = (\ker \varphi) \cap (R \cap R_i)S$, for some i. Again there is a factorization $\varphi: S \to S/K \twoheadrightarrow M$; and S/K is separated, by Lemma 2.1(ii). So $K = 0$, as desired.

(iii) $\Rightarrow$ (ii). First we note that $\ker(R \to \bar{R}) \ker \varphi = 0$: By (iii) each $(R \cap R_i) \ker \varphi \subseteq (R \cap R_i)S \cap \ker \varphi = 0$. Hence

$$\ker(R \to \bar{R}) \ker \varphi = \sum (R \cap R_i) \ker \varphi = 0.$$

Next we show that $\ker \varphi \subseteq \ker(R \to \bar{R})S$. Since $\ker(R \to \bar{R}) \ker \varphi = 0$, $\ker \varphi$ is a module over the semisimple artinian ring $\bar{R}$, and hence is a direct sum of simple $\bar{R}$-modules. Hence it suffices to show that every *simple* submodule U of $\ker \varphi$ is contained in $\ker(R \to \bar{R})S$. If not, then, since U is simple, $U \cap \ker(R \to \bar{R})S = 0$. Lemma 2.2 then shows that U is a direct summand of S, contrary to (iii). Thus $\ker \varphi \subseteq \ker(R \to \bar{R})S$.

(ii) $\Rightarrow$ (i). Here we are given $\varphi: S \twoheadrightarrow M$ satisfying (ii) and a factorization

$$\varphi: S \xrightarrow{f} S' \twoheadrightarrow M$$

with S' separated. We wish to show f is one-to-one. Take

$$s \in \ker f \subseteq \ker \varphi \subseteq \ker(R \to \bar{R})S = \sum_{i=1}^{n} (R \cap R_i)S,$$

say, $s = \sum s_i$ with $s_i \in (R \cap R_i)S$. Then $0 = f(s) = \sum f(s_i)$ with $f(s_i) \in (R \cap R_i)S'$. Directness of the sum $\sum (R \cap R_i)S'$ (Lemma 2.1) shows that each $f(s_i) = 0$ and hence $\varphi(s_i) = 0$. But φ is one-to-one on $(R \cap R_i)S$, by (ii); so each $s_i = 0$. Therefore $s = \sum s_i = 0$ as desired.

In the proof of (iii) $\Rightarrow$ (ii) we established the following fact, which will be needed again.

PROPOSITION 2.4. *In every separated representation* $\varphi: S \twoheadrightarrow M$, $\ker \varphi$ *is an* $\bar{R}$*-module via* $R \to \bar{R}$ (*that is*, $\ker(R \to \bar{R}) \ker \varphi = 0$).

PROPOSITION 2.5. *Let* $\varphi: S \twoheadrightarrow M$ *be a separated representation. Then no submodule of* S *other than* S *itself is mapped by* φ *onto* M.

Proof. Suppose $\varphi(T) = M$. Then $T + (\ker \varphi) = S$. Since $\ker \varphi$ is a module over the semisimple artinian ring $\bar{R}$, its submodule $T \cap \ker \varphi$ is a direct summand of it. Letting K be a complementary direct summand, we get $T \oplus K = S$. So by (iii) of Proposition 2.3, $K = 0$; in other words, $T = S$.

THEOREM 2.6 (The Almost Functorial Property). *Let* $\varphi': S' \twoheadrightarrow M'$ *and* $\varphi: S \twoheadrightarrow M$ *be separated representations. Then every* R*-homomorphism* $f: M' \to M$ *can be lifted to an* R*-homomorphism* $f^*: S' \to S$ *such that the following diagram commutes.*

$$\begin{tikzcd} S' \arrow[r, dashed, "f^*"] \arrow[d, two heads, "\varphi'"'] & S \arrow[d, two heads, "\varphi"] \\ M' \arrow[r, "f"] & M \end{tikzcd} \tag{1}$$

If f *is one-to-one or onto, so is every such* f^*.

Proof. Existence of f^*. Since φ is one-to-one on each $(R \cap R_i)S$ (by Proposition 2.3), we can define $f^*: (R \cap R_i)S' \to (R \cap R_i)S$ to be the composition

$$\varphi' \text{ then } f \text{ then } [(R \cap R_i)S \overset{\varphi}{\twoheadrightarrow} (R \cap R_i)M]^{-1}. \tag{2$_i$}$$

Moreover, since the sum $\sum (R \cap R_i)S'$ is *direct* (by Lemma 2.1), we can add the maps $(2)_i$ to get

$$f^*: \ker(R \to \bar{R})S' \to \ker(R \to \bar{R})S. \tag{3}$$

Now let T' be any submodule, $S' \supseteq T' \supseteq \ker(R \to \bar{R})\,S'$ to which the map f^* in (3) can be (further) extended. It will suffice to show that, if $T' \neq S'$, then f^* can be extended to a still larger submodule of S'; for then Zorn's lemma or transfinite induction immediately implies that f^* can be extended to all of S'.

Note that S'/T' is an $\bar{R}$-module, because $\ker(R \to \bar{R})\,S' \subseteq T'$. Since $S' \neq T'$, S'/T' has a nonzero cyclic submodule $\bar{R}(s' + T')$. And since $\bar{R}$ is semisimple artinian, we can choose s' such that, for some $\bar{e} = \bar{e}^2 \in \bar{R}$,

$$\bar{R}\bar{e} \cong \bar{R}(s' + T') \qquad \text{via} \qquad \bar{r}\bar{e} \leftrightarrow \bar{r}\bar{e}s' + T'. \tag{4}$$

Now choose any preimage e, in R, of $\bar{e}$ (caution: e might *not* be idempotent), and choose elements m', m, and s related as follows.

$$\begin{array}{ccc} s' & & s \\ {\scriptstyle\varphi'}\downarrow & & \downarrow{\scriptstyle\varphi} \\ m' & \xrightarrow{\;f\;} & m \end{array}\,. \tag{5}$$

We will extend f^* to $Res' + T'$ (which $\neq T'$ by (4), since $\bar{R}\bar{e} \neq 0$) by sending

$$res' + t' \to res + f^*(t') \qquad (r \in R,\, t' \in T'). \tag{6}$$

(Incidentally, specializing to the case $\bar{R} =$ a field so that we can take $e = 1$, we see, from the freedom which exists in the selection of s, that f^* is not always uniquely determined.)

To see that (6) is well-defined, it suffices to check that if res' belongs to T' (where f^* is already defined) then $f^*(res') = res$. But $res' \in T'$ implies that

$$0 = re(s' + T') = \bar{r}\bar{e}(s' + T') \qquad (\text{in } S'/T').$$

The isomorphism in (4) then shows that $\bar{r}\bar{e} = 0$; that is,

$$re \in \ker(R \to \bar{R}) = \sum (R \cap R_i).$$

Let $re = \sum r_i$ with $r_i \in R \cap R_i$. If we multiply (5) by r_i and then use the definition $(2)_i$ of f^* on $(R \cap R_i)\,S'$, we see that $f^*(r_i s') = r_i s$. Summing over i gives $f^*(res') = res$, completing the proof that f^* exists.

Now *suppose f is onto*. Then, by commutativity of the square (1), $f^*(S')$ is a submodule of S which φ maps *onto* M. By Proposition 2.5, $f^*(S') = S$; that is, f^* is onto.

Finally, *suppose* f is one-to-one. Then $f\varphi': S' \to f(M')$ is a separated representation of $f(M')$, and has a factorization

$$f\varphi': S' \xrightarrow{f^*} S \overset{\varphi}{\twoheadrightarrow} f(M').$$

Since S is separated, we conclude that f^* is one-to-one. Q.E.D.

The next corollary states that separated representations can be found wherever you might be tempted to look for them!

COROLLARY 2.7. *Let $\varphi': S' \twoheadrightarrow M$ be an epimorphism of R-modules, with S' separated; and let $\varphi: S \twoheadrightarrow M$ be a separated representation. Then there is a factorization*

$$\varphi': S' \twoheadrightarrow S \overset{\varphi}{\twoheadrightarrow} M.$$

Proof. A careful reading of the proof of Theorem 2.6 would show that this corollary has already been proved. However, the corollary can be recovered less painfully by first noting that the identity map: $S' \to S'$ is a separated representation, and then applying the theorem to the diagram

$$\begin{tikzcd} S' \arrow[r, dashed, "f^*"] \arrow[d, two heads, "id"'] & S \arrow[d, two heads, "\varphi"] \\ S' \arrow[r, two heads, "\varphi'"] & M \end{tikzcd}$$

THEOREM 2.8. *Every R-module M has a separated representation $\varphi: S \twoheadrightarrow M$. If $\varphi': S' \twoheadrightarrow M$ is another separated representation of M, then there is an isomorphism f^* of S' onto S such that $\varphi f^* = \varphi'$.*

Proof. The uniqueness is the special case of the almost functorial property obtained by taking f to be the identity map on M.

For the existence part, take any epimorphism $\varphi': F \twoheadrightarrow M$ with F separated. For example, F can be any sufficiently large free module.

The idea for the proof comes from Corollary 2.7 which states that *if M* has a separated representation φ, then φ' can be factored through φ.

So consider the family $\mathscr{F}$ of submodules H of F such that

$$H \subseteq \ker \varphi', \tag{7}$$

$$\bigcap_{i=1}^{n} [\ker(R \to R_i)(F/H)] = 0. \tag{8}$$

This family is nonempty since it contains 0 (Lemma 2.9 below) and is partially ordered by inclusion. A straightforward use of Zorn's lemma shows that $\mathscr{F}$ contains a maximal element H.

Let $\varphi\colon (F/H) \twoheadrightarrow M$ be the map obtained by factoring φ' through F/H. Then φ is a separated representation because (8) and Lemma 2.9 below show that F/H is separated and because maximality of H provides the "as close as possible to M" property.

LEMMA 2.9. *An R-module S is separated $\Leftrightarrow$*

$$\bigcap_{i=1}^{n} \ker(R \to R_i)S = 0. \tag{9}$$

Proof. ($\Rightarrow$) If $S \subseteq S_1 \oplus \cdots \oplus S_n$ with each S_i an R_i-module, then (9) clearly holds.

($\Leftarrow$) By (9) there is a monomorphism of S into $\oplus_{i=1}^{n} S_i$, where $S_i = S/\ker(R \to R_i)S$. By construction S_i is an R_i-module.

COROLLARY 2.10. *Let $\varphi\colon S \twoheadrightarrow M$ be a separated representation. If M is finitely generated, so is S.*

Proof. Let $s_1,\dots, s_n$ be preimages, in S, of a finite set of generators of M. Then $\varphi(\sum Rs_i) = M$; so, by minimality property (2.5), $\sum Rs_i = S$.

Remark 2.11. It might be of interest to a user of the results of this section that when each R_i is a prime ring and each $R \cap R_i \neq 0$, the subdirect sum decomposition of R used above is unique. In particular, $\bar{R}$ and each R_i are determined up to isomorphism by R. For details, see [2].

3. PULLBACKS AND MULTIPLE PULLBACKS

Our object is to show that the rings and separated modules of the preceding two sections are pullbacks when $n = 2$. When $n > 2$, we show that R is a "multiple pullback."

"DIAMOND" LEMMA 3.1. *In the commutative diagram of abelian groups and homomorphisms shown in Fig. 1, the following conditions are equivalent.*

(i) *Whenever $f_1(s_1) = f_2(s_2)$ there is an s in S such that $\pi_1(s) = s_1$ and $\pi_2(s) = s_2$.*

(ii) $\ker(S \to S_1) + \ker(S \to S_2) = \ker(S \to \bar{S})$.

$$\begin{tikzcd}
& S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\theta"] & \\
S_1 \arrow[dr, "f_1"'] & & S_2 \arrow[dl, "f_2"] \\
& \bar{S} &
\end{tikzcd}$$

FIGURE 1

We omit the straightforward proof. Note that every pullback of $S_1 \oplus S_2$ is a subdirect sum. One immediate consequence of the above lemma is the well-known fact that every subdirect sum of *two* groups is a pullback:

PROPOSITION 3.2. *Let S be a subdirect sum of $S_1 \oplus S_2$, with projection maps $\pi_i: S \twoheadrightarrow S_i$. Let θ be the natural homomorphism of S onto $\bar{S} = S/(\ker \pi_1 + \ker \pi_2)$, and let f_1 and f_2 be the maps that make Fig. 1 commute. Then S is the pullback of $S_1 \oplus S_2$ given by*

$$S = \{(s_1, s_2) \in S_1 \oplus S_2 \mid f_1(s_1) = f_2(s_2)\}. \qquad (1)$$

Proof. The inclusion $S \subseteq \{\cdots\}$ is obvious, while the opposite inclusion follows from Lemma 3.1.

COROLLARY 3.3. *Let the ring R be the pullback of $R_1 \oplus R_2$ determined by Fig. 2, that is, $R = \{(r_1, r_2) \in R_1 \oplus R_2 \mid \nu_1(r_1) = \nu_2(r_2)\}$. Then each separated R-module S is a pullback, as in (1) above, of (an R_1-module S_1) $\oplus$ (an R_2-module S_2) combined by an $\bar{R}$-module $\bar{S}$. The representation (1) can be chosen "canonically" with $\ker(S_i \to \bar{S}) = \ker(R_i \to \bar{R})\, S_i$.*

$$\begin{tikzcd}
& R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd] & \\
R_1 \arrow[dr, "\nu_1"'] & & R_2 \arrow[dl, "\nu_2"] \\
& \bar{R} &
\end{tikzcd}$$

FIGURE 2

Proof. Let $S_i = S/\ker(R \to R_i)S$ so that S_i is an R_i-module. By Lemma 2.9, we can consider $S \subseteq S_1 \oplus S_2$. Call the projection maps π_i. Then Proposition 3.2 states that S is given by (1) above. We next prove that $\bar{S}$ is an $\bar{R}$-module, that is, $\ker(R \to \bar{R})\bar{S} = 0$.

By the Diamond Lemma, applied to Fig. 2,

$$\ker(R \to \bar{R}) = \ker(R \to R_1) + \ker(R \to R_2).$$

Multiplying this by S we get

$$\ker(R \to \bar{R})S = \ker(S \to S_1) + \ker(S \to S_2) = \ker(S \to \bar{S}) \tag{2}$$

so $\ker(R \to \bar{R})\bar{S} = 0$ as desired. Finally, in the notation of Fig. 1,

$$\begin{aligned}
\ker(S_i \to \bar{S}) &= \pi_i(\ker(S \to \bar{S})) \\
&= \pi_i(\ker(R \to \bar{R})S) \qquad \text{by (2)} \\
&= \ker(R \to \bar{R})\, S_i = [(R \cap R_1) + (R \cap R_2)]\, S_i. \\
&= (R \cap R_i)\, S_i = \ker(R_i \to \bar{R})\, S_i.
\end{aligned}$$

This completes the case $n = 2$. The case $n > 2$, which follows, will not be used in [1].

Multiple Pullbacks

Let rings $R_1, \ldots, R_n$ be given, and let a finite number of *pairs* of *onto* homomorphisms be given:

$$f_{k,i(k)}\colon R_{i(k)} \twoheadrightarrow C(k) \qquad \text{and} \qquad f_{k,j(k)}\colon R_{j(k)} \twoheadrightarrow C(k), \qquad \text{with} \qquad i(k) < j(k).$$

One such pair is shown in $(3)_k$ below.

$$\begin{tikzcd}[column sep=small]
R_1 \cdots & R_{i(k)} \arrow[dr, "f_{k,i(k)}"'] & \cdots & R_{j(k)} \arrow[dl, "f_{k,j(k)}"] & \cdots R_n \quad (i(k) < j(k)) \\
& & C(k) & &
\end{tikzcd} \tag{3_k}$$

We define the *multiple pullback* of $R_1 \oplus \cdots \oplus R_n$ determined by these *combining homomorphisms* to be

$$R = \left\{ (r_1, \ldots, r_n) \in \bigoplus_{i=1}^{n} R_i \mid (\forall k)\, f_{i(k)}(r_{i(k)}) = f_{j(k)}(r_{j(k)}) \right\}. \tag{4}$$

We call the rings $C(k)$ the *combining rings* of the multiple pullback (4). When $n = 2$, the multiple pullback coincides with the ordinary pullback.

Note that no additional generality is gained by allowing combining homomorphisms to involve three (or more) of the coordinate rings, because a statement, such as $f_1(r_1)=f_2(r_2)=f_3(r_3)$ in (4) would define the same R as the pair of statements $f_1(r_1)=f_2(r_2)$ and $f_2(r_2)=f_3(r_3)$.

THEOREM 3.4. *Let the ring R be a subdirect sum of $R_1 \oplus \cdots \oplus R_n$. Then* (i) *implies* (ii).

(i) $\bar{R}=R/\oplus_{i=1}^{n}(R \cap R_i)$ *is semisimple artinian.*

(ii) *R is a multiple pullback of $R_1 \oplus \cdots \oplus R_n$ whose combining rings $C(k)$ are all semisimple artinian.*

The proof will require:

LEMMA 3.5. *Let the ring R be a subdirect sum of $R_1 \oplus \cdots \oplus R_n$.*

(i) *If M is a two-sided* (*or left or right*) *ideal of R whose ith projection M_i equals R_i for every i, then $M=R$.*

(ii) *If M is a maximal two-sided* (*or left or right*) *ideal and j is an index such that $M_j \neq R_j$, then M_j is a maximal ideal of R_j, and $M=\{r \in R \mid r_j \in M_j\}$.*

Proof. (i) By letting R' = the projection of R in $R_2 \oplus \cdots \oplus R_n$ we see that R is a subdirect sum of $R_1 \oplus R'$ and get a reduction to the case $n=2$.

Since $M_1=R_1$ and $M_2=R_2$, M contains elements of the forms $(1, b)$ and $(a, 1)$. Hence it also contains $(1, b)(a, 1)=(a, b)$; so it also contains $(1, b)+(a, 1)-(a, b)=(1, 1)$. Thus $M=R$.

For (ii), note that such a j exists by (i). Then $M_j \subseteq$ some maximal ideal N_j of R_j. Then $N=\{r \in R \mid r_j \in N_j\}$ is an ideal $\neq R$ and contains M. By maximality of M, $M=N$, as desired.

Proof of the theorem. In preparation for a proof by induction on n, let R' be the projection of R in $R_2 \oplus \cdots \oplus R_n$. We note first that $R'/\oplus_2^n(R' \cap R_i)$ *is semisimple artinian*: Since the projection map $R \twoheadrightarrow R'$ takes $R \cap R_i$ *into* $R' \cap R_i$ ($i \geqslant 2$) we see that the semisimple artinian ring $\bar{R}=R/\oplus(R \cap R_i)$ can be mapped *onto* $R'/\oplus(R' \cap R_i)$. Hence this latter ring is also semisimple artinian.

Since R is a subdirect sum of $R_1 \oplus R'$, the case $n=2$ (Proposition 3.2) shows that R is a pullback of $R_1 \oplus R'$, say

$$R=\{(r_1, r') \in R_1 \oplus R' \mid f_1(r_1)=f'(r')\}, \tag{5}$$

where f_1 and f' map each of R_1 an R' onto, say, C.

We claim that C *is semisimple artinian.* Referring to Fig. 2, with $R_2 = R'$ and $\bar{R} = C$, we see that

$$C \cong R_1/\ker v_1 = R_1/(R \cap R_1).$$

Since R can be projected onto R_1 and then mapped onto C, as above, and since in this composite map the kernel $\oplus_{i=1}^{n}(R \cap R_i)$ of $R \twoheadrightarrow \bar{R}$ is sent to zero, we conclude that $\bar{R}$ can be mapped onto C. Since $\bar{R}$ is semisimple artinian, so therefore is C.

Now, by the first paragraph of this proof (and induction), R' is a multiple pullback of $R_2 \oplus \cdots \oplus R_n$ with semisimple artinian combining rings. To complete the proof we have to replace the condition $f_1(r_1) = f'(r')$ in (5) by a set of conditions, each involving only two coordinates R_1 and R_j ($j \geqslant 2$).

Since C is semisimple artinian, we can write $C = \oplus_{i=1}^{t} C(k)$ with each $C(k)$ simple artinian. Then the condition $f_1(r_1) = f'(r')$ can be replaced by the t conditions

$$f_{1k}(r_1) = f'_k(r'), \tag{6}$$

where subscript k indicates composition with the projection map $C \twoheadrightarrow C(k)$. For each k, $\ker(f'_k: R' \twoheadrightarrow C(k))$ is a maximal ideal of R'. Statement (ii) of the lemma provides an index $j \geqslant 2$ such that f'_k has a factorization

$$f'_k: R' \twoheadrightarrow R_j \xrightarrow{g_j} C(k).$$

To complete the proof we merely replace (6) by

$$f_{1k}(r_1) = g_j(r_j)$$

and do the same for each k.

References

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